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TURBULENT SCHMIDT NUMBERS

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Abstract—Eddy diffusivities of mass were determined using a point source technique. The turbulent diffusion of helium, carbon dioxide and n-octane into air was studied. Results are correlated in terms of the turbulent Schmidt number as a function of eddy viscosity and molecular Schmidt number.

NOTATION

A, constant, $= Q/4\pi E_D X$;
$\bar{C}$, time mean concentration;
D, molecular diffusivity;
E_D, total diffusivity, $E_D = D + \epsilon_D$;
Q, source gas rate;
R, duct radius;
r, radial distance from sample point to point of maximum concentration;
$\bar{U}$, time mean axial velocity;
X, distance along the axial direction;
$\bar{Y}^2$, variance;
y, distance along the radial direction.

Dimensionless groups

N_{Pr}, Prandtl number;
N_{Re}, Reynolds number;
N_{Sc}, Schmidt number.

Greek symbols

α, thermal diffusivity;
ϵ_α, eddy diffusivity of heat;
ϵ_D, eddy diffusivity of mass;
ϵ_ν, eddy diffusivity of momentum;
ϵ_ν/ϵ_D, turbulent Schmidt number;
ϵ_ν/ν, relative viscosity;
ν, kinematic viscosity.

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RECENT workers [1, 2] have introduced the turbulent Schmidt or Prandtl number as a parameter in their calculations of turbulent transport. This parameter is the ratio of the eddy viscosity, ϵ_ν, to the eddy diffusivity of mass, ϵ_D, or heat, ϵ_α. While earlier investigators have assumed that the turbulent Prandtl and Schmidt numbers were unity, it has become apparent in recent years that they are functions of the nature of the flow and of molecular properties.

The purpose of this study was to investigate the effect of molecular Schmidt number and turbulence level on the eddy diffusivity of mass.

Background

The major portion of the work in turbulent transport has dealt with the transmission of momentum and vorticity. Deissler [3] has developed a modification of the universal velocity distribution for flow in conduits. His results apply in both the wall region and the central core. More recent modifications [4] have eliminated discontinuities in the velocity and eddy viscosity profiles.

These results have been used in analogy calculations of the temperature profile and heat transfer coefficient. In most cases [3] it has been assumed that the turbulent Prandtl number was unity. However, recently several investigators have removed this assumption by employing

empirical or semitheoretical values for the turbulent Prandtl number [5–8]. The values of the turbulent Prandtl number used by these investigators have for the most part been obtained by differentiating measured profiles in air, water, and mercury.

Theoretical investigations of the Turbulent Prandtl and Schmidt numbers have been attempted. Using a modification of Prandtl's mixing length, Jenkins [9] and Azer and Chao [6] and others have related the Turbulent Prandtl number to the molecular Prandtl number and the eddy viscosity. Marchello and Toor [2] employed penetration theory methods to develop a mixing layer model for the low level turbulence near a phase boundary. While the numerical predictions of the various models differ greatly, they both predict that for heat transfer

$$\frac{\epsilon_v}{\epsilon_\alpha} = f\left(\frac{\epsilon_v}{\nu}, N_{Pr}\right) \tag{1a}$$

and for mass transfer

$$\frac{\epsilon_v}{\epsilon_D} = f\left(\frac{\epsilon_v}{\nu}, N_{Sc}\right). \tag{1b}$$

Agreement between the model prediction and experimental data has been achieved by combining them [2] or by adjusting them to fit ϵ_α data for air [7]. Both models predict that the turbulent Prandtl and Schmidt numbers are unity when their molecular counterparts are.

EQUIPMENT AND PROCEDURE

The eddy diffusivity measurements were made in a copper pipe having an inside diameter of 3·43 in. Filtered air was supplied by a blower and passed through a surge tank prior to entering the pipe. The test section of pipe, containing the source and sampling probes, in parallel, was installed fifty diameters from the air inlet. After leaving the test section, the air passed through a Rotameter and was exhausted.

The source probe was a stainless steel capillary tube having an outside diameter of 0·060 in. and an inside diameter of 0·018 in. The sampling probe was a piece of hypodermic needle tubing having an outside diameter of 0·0356 in. This probe was sealed at one end, and a small hole, 0·016 in. dia., was drilled into its side. Sculpture clay was used to seal around the probes at the points where they entered the pipe wall.

The probes were attached to micrometer positioning mechanisms [13] which allowed them to be set at any desired radial position with a precision of 0·001 in. The source probe was straight and has a rounded tip. It was located at a fixed axial position 50 diameters downstream. The thinner sample probe passed completely through the pipe for support and was held so that the sample area was always perpendicular to the pipe axis. This probe could be located at any one of five different axial positions downstream of the source probe. These were 0·0751, 0·500, 0·999, 1·999 and 5·014 in. respectively from the source probe location.

Helium and carbon dioxide were fed from gas cylinders and passed through small gas rotameters prior to entering the source probe. For the n-octane studies nitrogen was used as a carrier gas. The nitrogen was bubbled through the liquid octane, and the resulting mixture passed through a Rotameter and into the turbulent air stream via the source probe.

Gas chromatography was used for the analysis of all gas samples [13]. A glass tee was placed in the line leading from the sample probe. One leg of the tee was fitted with a rubber septum through which samples for the chromatograph were obtained. A similar arrangement was used to get source samples when the n-octane was studied.

Before carrying out the diffusion studies, the air velocity profile was measured with the source probe at various radial positions. For these measurements a displacement type micromanometer was employed and the sample probe was used as an impact tube [13]. To employ the straight sampling probe as an impact tube, it

was necessary to account for the flow of air over the probe surface. This analysis yielded a method for determining the effect of impact tube size [14].

The results of the velocity studies indicated that the generalized velocity distribution could be used to describe the fluid velocity in the region between the source and sample points [13]. In addition, friction factor measurements immediately in front of the test section agreed with existing correlations for fully developed turbulent flow.

EDDY DIFFUSIVITIES

For steady state diffusion from a point source into a turbulent flow field the diffusion equation in cylindrical coordinates, assuming angular symmetry, is

$$\bar{U}\frac{\partial \bar{C}}{\partial X} = E_D\left(\frac{\partial^2 \bar{C}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{C}}{\partial r}\right) \tag{2}$$

where $E_D = D + \epsilon_D$ and $\bar{C}$ and $\bar{U}$ are the time average values of the concentration and axial fluid velocity. In equation (2) axial diffusion is assumed to be negligible, and E_D is assumed constant. This equation is strictly true only for small radial distances between source and sample points. Also, when the source is near the wall the turbulence is not isotropic and the diffusion is not axially symmetric.

The turbulent spread of matter from a point source follows a Gaussian distribution (10, 11) over fairly short distances and times during which the flow field appears homogeneous and infinite in extent. Thus the solution to equation (2) was chosen to be:

$$\bar{C} = A\exp(-r^2/2\bar{Y}^2) \tag{3}$$

where $\bar{Y}^2$ is the variance which serves as a measure of the spread of material and A is determined from the source boundary conditions. The variance may be expressed as [10, 12, 13]

$$\bar{Y}^2 = \frac{2E_D X}{\bar{U}}. \tag{4}$$

The constant A is evaluated in terms of the source strength from the material balance at any axial position

$$Q = 2\pi\int_0^\infty \bar{C}\,\bar{U}\,r\,dr \tag{5}$$

where Q is the source strength. The upper integration limit assumes that the spread of material occurs over a short radial distance compared to the radius of the duct.

From equations (3), (4) and (5) the final form of the solution is

$$\bar{C} = \frac{Q}{4\pi E_D X}\exp\left[-\left(\frac{\bar{U}r^2}{4E_D X}\right)\right]. \tag{6}$$

Taking the natural logarithm

$$\ln \bar{C} = \ln\frac{Q}{4\pi X} - \ln E_D - \frac{\bar{U}r^2}{4E_D X}. \tag{7}$$

With the exception of E_D all the other quantities in equations (6) and (7) are known or can be measured. Thus equation (7) was used to obtain E_D from the experimental data.

The eddy diffusivity was determined from a least squares fit of the concentration data to equation (7). The per cent standard deviation between the experimental values and the computed line was always less than 5 per cent [13]. Velocity data indicated that the generalized velocity correlation could be used to obtain point velocities in this region. In applying equation (7), it was assumed that the fluid velocity and eddy diffusivity were constant and equal to their average values over the small radial spread of the material, r, between the source and sample points.

Comparison of the experimental results with other investigators work poses several difficulties. Most of the existing data on eddy diffusivity of mass has been taken at the center

of the pipe. The data of several investigators are shown in Fig. 1. Towle and Sherwood [16] and Flint, Kada and Hanratty [17] have determined eddy diffusivities for both hydrogen and carbon

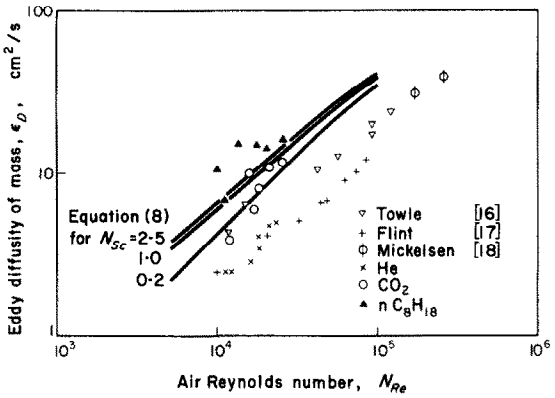

FIG. 1. Eddy mass diffusivity in central pipe core.

dioxide diffusing in air. These investigators found no experimental difference in the eddy coefficients for hydrogen and carbon dioxide. The data of Mickelson [18] for helium in air and the results of the present study for helium, carbon dioxide and normal octane are also presented in Fig. 1. All the data presented in Fig. 1 were taken at the pipe center.

Azer and Chao [6] surveyed the existing eddy diffusivity of heat data and using a modified mixing-length model proposed the following correlation:

$$\frac{\epsilon_\alpha}{\epsilon_v} = \left\{ \frac{1 + 135\, N_{Re}^{-0.45} \exp[(Y/R)^{0.25}]}{1 + 57\, N_{Re}^{-0.46}\, N_{Pr}^{-0.58} \exp[-(Y/R)^{0.25}]} \right\}. \quad (8)$$

The authors state that equation (8) fits the available data in the range $0.6 \lesssim N_{Pr} \lesssim 15$ with a maximum deviation of less than 14 per cent.

The mass transfer equivalent of equation (8) for molecular Schmidt numbers of 0·2, 1·0 and 2·5 is plotted on Fig. 1 for the central pipe core, $Y = R$. This was accomplished by replacing ϵ_α and N_{Pr} with ϵ_D and N_{Sc} respectively. ϵ_v was obtained from Deissler's correlation [3]. While the curves for equation (8) group rather closely, they do show the same general dependency on molecular properties as the data of this work.

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The eddy diffusivities and the corresponding turbulent Prandtl and Schmidt numbers vary as illustrated in equation (8). The data taken in the isotropic core region shown in Fig. 1, do not reflect this variation.

Since the eddy diffusivity of momentum increases with distance from the wall either variable may be used to represent the data, equation (1) and (8). Figures 2, 3 and 4 present the turbulent Schmidt number as a function of the ratio of the turbulent to the molecular kinematic viscosities for helium, carbon dioxide

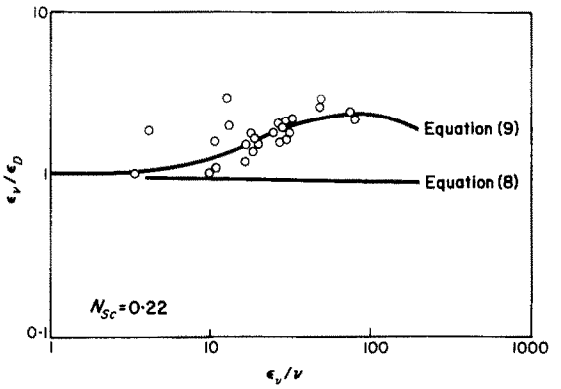

FIG. 2. Turbulent Schmidt number of helium.

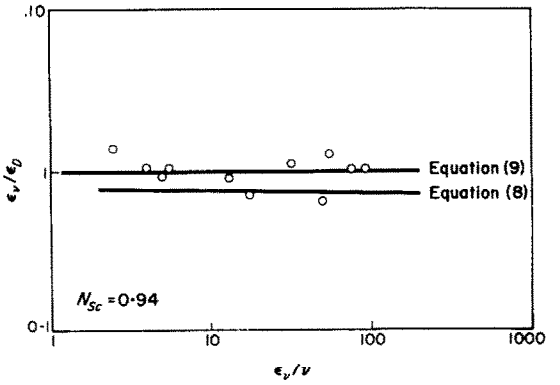

FIG. 3. Turbulent Schmidt number of carbon dioxide.

and normal octane diffusing in air. These data cover a range of $0{\cdot}05 \lesssim Y/R \lesssim 0{\cdot}8$ and $5000 \lesssim N_{Re} \lesssim 30{,}000$. While the data scatter quite widely it does indicate that the conclusions of equation (1) may be valid.

The predictions of equation (8) for a Reynolds number of 20,000 are also shown on Figs. 2, 3 and 4. This was accomplished in the same

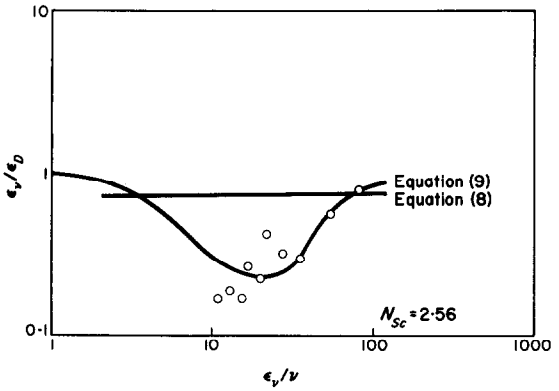

FIG. 4. Turbulent Schmidt number of *n*-octane.

manner as for Fig. 1. In an effort to represent the data and models more closely than equation (8), the following correlation was developed [13]:

$$\frac{\epsilon_v}{\epsilon_D} = \left\{1+\frac{\epsilon_v}{\nu}\frac{1}{(N_{Sc}+2{\cdot}93)} - 0{\cdot}254 \times \exp\left[-\left(\frac{\epsilon_v}{\nu}\right)[0{\cdot}015\,N_{Sc} + 0{\cdot}00769]\right]\right\}. \quad (9)$$

Only the data obtained in this study were used in developing equation (9).

Curves for equation (9) are presented in Figs. 2, 3 and 4. This equation is empirical but does use the variables predicted from the Jenkins and the mixing models. It also possesses the limits of these models for both low and high level turbulence and when the molecular Schmidt number is unity. While not plotted in Figure 1 to avoid confusion, equation (9) shows a wider spread with Schmidt number than equation (8) and passes through the data for helium, carbon dioxide and normal octane.

Several investigations have used wetted wall columns [19] and similar devices to measure eddy mass diffusivities. In these instances the diffusion of water vapor or carbon dioxide was studied. The results agree with Fig. 3 in that ϵ_D is nearly coincident with ϵ_D across the radius. The same conclusion has been reached for the large amount of data on heating and cooling of air [6, 13]. The molecular Prandtl number is 0·7 and ϵ_α data are roughly equal to ϵ_v.

Analogy calculations were performed employing the values of the turbulent Schmidt or Prandtl number predicted by equation (9). The computed Sherwood or Nusselt numbers showed good agreement with the empirical correlations representing heat and mass transfer data over the range from 0·01 to 1000 on the molecular Prandtl or Schmidt numbers [13]. Azer and Chao also found good agreement in their analogy calculations using equation (8). These conclusions simply illustrate the fact that the smoothing effect of the analogy integration obscures the differences between equations (8) and (9) and consequently analogy calculations do not provide a rigorous test of eddy diffusion data.

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Résumé—Les diffusivités turbulentes massiques ont été déterminées en utilisant une technique de source ponctuelle. La diffusion turbulente de l'hélium, du gaz carbonique et du n-octane dans l'air a été étudiée. Les résultats sont corrélés à l'aide du nombre de Schmidt turbulent en fonction de la viscosité turbulente et du nombre de Schmidt moléculaire.

Zusammenfassung—Die Koeffizienten des turbulenten Stoffaustausches wurden nach der Punktquellen-Technik berechnet. Die turbulente Diffusion von Helium, Kohlendioxid und n-Oktan in Luft wurde untersucht. Die Ergebnisse werden in Form der turbulenten Schmidt-Zahl korreliert als Funktion der Scheinzähigkeit und molekularen Schmidt-Zahl.

Аннотация—Коэффициенты турбулентной диффузии массы определялись с помощью метода точечного источника. Изучалась турбулентная диффузия в воздух гелия, двуокиси углерода и *n*-октана. Результаты обобщаются с помощью турбулентного критерия Шмидта как функции турбулентной вязкости и молекулярного критерия Шмидта.